Novel Series-based Approximations to e

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In this paper we dare to take one of the oldest dogs in the college calculus curriculum—the Taylor series—and teach *you* how to do new tricks with it that Newton, Euler and their successors do not seem to have discovered. Below, we use this standard tool of introductory-level calculus to derive very accurate closed-form approximations to e. The expressions we derive here and in a companion paper [**2**] appear to be new, even though approximations to e were first discovered in the 1600's [**3**, pp. 26-27]. Using our technique, you and your students can—with a little perseverance—be able to derive and prove for yourselves entirely new and highly accurate methods of calculating e.

An old stand-by of college calculus textbooks [**1**, p. 558; **4**, p. 743] is the calculation of e via $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$:

$$\textit{Classical}:\quad \left(1+\frac{1}{x}\right)^x \approx e. \tag{1}$$

For example, inserting $x = 1000$ into (1) we get 2.71692 39322, which is e accurate to two decimal places.

Elsewhere in most college calculus textbooks [e.g., **1**, p. 654; **4**, p. 711] e is obtained directly from the Maclaurin series for e^x, which is $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$. For $x = 1$ this equals

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{N!} \approx e.$$

Although it is almost never done in calculus textbooks, it is a relatively simple task to express the Classical approximation as a series. We can then evaluate the error of (1) analytically rather than by trial-and-error.

First, express $\ln(1+x)$ as a Maclaurin series, convergent for $-1 < x \le 1$ [**1**, p. 635]:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots$$

Next, replace x with $\frac{1}{x}$ and multiply the result by x:

$$\begin{aligned} x\ln\left(1+\frac{1}{x}\right) &= \ln\left(1+\frac{1}{x}\right)^x \\ &= 1 - \frac{1}{2x} + \frac{1}{3x^2} - \frac{1}{4x^3} + \frac{1}{5x^4} - \frac{1}{6x^5} + \frac{1}{7x^6} - \cdots \equiv p(x),\ x \ge 1. \end{aligned}$$

Thus

$$\begin{aligned} \left(1+\frac{1}{x}\right)^x &= e^{p(x)} = e \cdot e^{p(x)-1} \\ &= e\left[1 + (p(x)-1) + \frac{(p(x)-1)^2}{2!} + \frac{(p(x)-1)^3}{3!} + \ \ldots\right] \end{aligned}$$

or

$$\left(1+\frac{1}{x}\right)^x = e\left[1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238\,043}{580\,608x^6} - \ldots\right],\quad x \ge 1. \qquad (2)$$

The series in (2) shows the error in the Classical approximation. The approximation is not very accurate for small x because it is only first-order accurate—that is, the series in (2) possesses a term that is proportional to $1/x$. This term is relatively large for small x. Since the series for the Classical approximation indicates where its Achilles' heel is in terms of accuracy, we can use this information to create new algebraic expressions that improve upon its accuracy.

The path to obtaining new and more accurate approximations to e using series involves repeated "bootstrapping," i.e., combining two good approximations to get a better one. We first obtain an approximation of e accurate to second order, that is, of the form $e\left[1 + O\left(\frac{1}{x^2}\right)\right]$.

We can derive a second-order approximation to e by summing the two series for $x\ln\left(1+\frac{1}{x}\right)$ and $\ln\left(1+\frac{1}{2x}\right)$:

$$\begin{aligned} x\ln\left(1+\frac{1}{x}\right) + \ln\left(1+\frac{1}{2x}\right) &= \left(1 - \frac{1}{2x} + \frac{1}{3x^2} - \frac{1}{4x^3} + \ldots\right) \\ &\quad + \left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} - \ldots\right) \\ &= 1 + \frac{5}{24x^2} - \frac{5}{24x^3} + \ldots \equiv q(x). \end{aligned}$$

Thus

$$\left(1+\frac{1}{x}\right)^x\left(1+\frac{1}{2x}\right)=e\cdot e^{q(x)-1}$$

or

$$ACM:\ \left(1+\frac{1}{x}\right)^x\left(1+\frac{1}{2x}\right)$$
$$=e\left[1+\frac{5}{24x^2}-\frac{5}{24x^3}+\frac{1187}{5760x^4}-\frac{587}{2880x^5}+\frac{117\,209}{580\,608x^6}-\ldots\right],\ x\geq 1. \qquad (3)$$

We call this the "Accelerated Classical Method," or ACM. Here and later, the acronym refers to the closed-form expression on the left-hand side of the equation.

The series in (3) shows that ACM is more accurate than the Classical approximation, because the power of x in the first term is larger; therefore the sum of the terms in the brackets on the right-hand side of (3) is closer to 1 for all $x > 1$ than in the case of (2). However, every math classroom has its "Doubting Thomas" who must test the theory for himself or herself. Fortunately, comparison of these approximations is easy with a calculator. ACM is equal to the Classical approximation multiplied by $\left(1+\frac{1}{2x}\right)$. Therefore, for $x = 1000$ ACM is equal to $2.71692\,39322 \times 1.0005 = 2.71828\,23942$, which is e accurate to *six* decimal places. The calculator confirms the calculus; ACM is superior to the Classical approximation!

(ACM was originally derived by numerically examining the relationship between the Classical approximation and another new method. See Appendix A for this alternative derivation, which includes additional material of use in introductory calculus courses.)

Another second-order-accurate approximation to e results when we add the series for $x\ln\left(1+\frac{1}{x}\right)$ to its "mirror image" with x replaced by $-x$:

$$-x\ln\left(1-\frac{1}{x}\right)=1+\frac{1}{2x}+\frac{1}{3x^2}+\frac{1}{4x^3}+\frac{1}{5x^4}+\frac{1}{6x^5}+\frac{1}{7x^6}+\cdots.$$

The $\frac{1}{2x}$ terms cancel in this summation, as do all the odd-powered terms in x. Dividing the sum by 2 and exponentiating yields the "Mirror Image Method," or MIM:

$$MIM:\ \left(\frac{x+1}{x-1}\right)^{\frac{x}{2}}=e\left[1+\frac{1}{3x^2}+\frac{23}{90x^4}+\frac{1223}{5670x^6}+\ldots\right],\ x>1. \qquad (4)$$

See [**2**] for more discussion of this method and its numerous extensions.

Now that we have two second-order-accurate methods for approximating e, we can combine them to create a third-order-accurate approximation by summing the series for ACM with $-\frac{5}{8}\times$ the series for MIM, and then multiplying this sum by $\frac{8}{3}$.

The result is

$$ACMMIM:\ (x+1)^{\frac{11x}{6}}(x-1)^{\frac{5x}{6}}\left(\frac{2x+1}{2x^{x+1}}\right)^{\frac{8}{3}}$$
$$=e\left[1-\frac{5}{9x^3}+\frac{19}{120x^4}-\frac{77}{180x^5}+\frac{137}{1008x^6}-\ldots\right],\ x>1. \qquad (5)$$

In Figure 1, we present a visual comparison of ACM, MIM, and ACMMIM with the Classical approximation. [We emphasize for clarity that the *left*-hand sides of (3), (4), and (5) are used in these calculations; the right-hand sides of these equations are simply the analytical machinery that guided the creation of the closed-form expressions.] The figure demonstrates that all three of these new methods significantly improve upon the Classic closed-form expression.

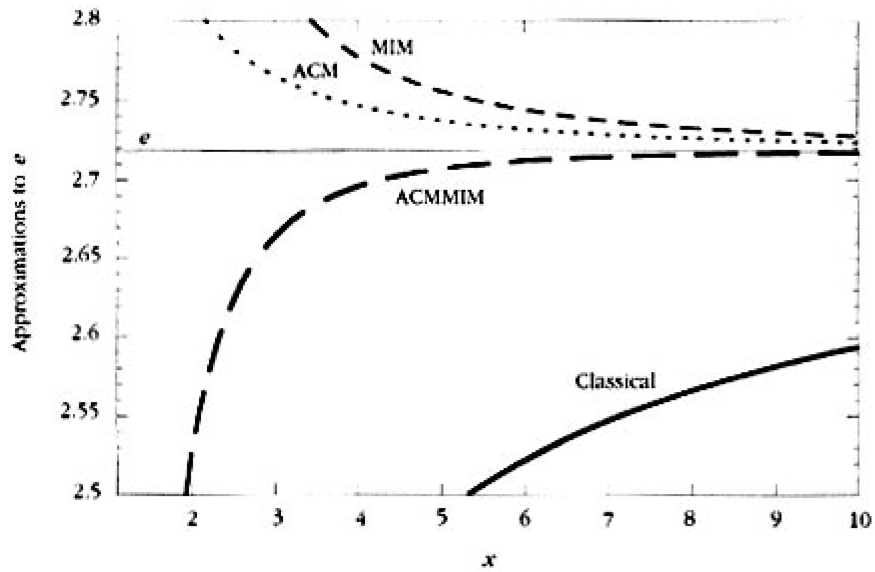

Figure 1. A comparison of the new approximations ACM, MIM, and ACMMIM versus the Classical approximation to e for $1 < x \leq 10$.

To summarize, our "bootstrapping" approach for deriving highly accurate closed-form approximations to e is composed of three steps:

Step 1: Take the series for two (or more) algebraic expressions and add them together in such a way that the lowest-power term in x in the sum of the series cancels out.

Step 2: Multiply both sides of this sum—the sum of the algebraic expressions on the left-hand side, the sum of their series on the right-hand side—by a constant. This coefficient (e.g., $8/3$ in ACMMIM) is dictated by what you need to multiply the right-hand side with so that its constant term is exactly 1. Why do this step? Because in Step 3, you will exponentiate both sides of the equation. If the constant term is 1, then in Step 3 you will obtain $e^{1 + \text{small terms in } x} \approx e$, and that's what we're after!

Step 3: Exponentiate both sides to get the final result. The left-hand side gives you a closed-form algebraic expression that is a very accurate approximation to e, and the right-hand side gives you a calculus-based quantification of how good this approximation is.

We believe that many—one would hope most—college calculus students can grasp these simple manipulations. Furthermore, approximations much more accurate than ACMMIM can be obtained through this approach. For example, in Appendix B we cite without derivation several new approximations to e obtainable via the approach outlined in this paper. In [**2**], we discuss how more exotic variants of MIM can lead to exceptionally accurate approximations.

In conclusion, how often are you able to teach a 1990's-vintage research result in introductory calculus? We encourage teachers to present our results in the classroom when you discuss the compound-interest formula, or when you introduce Taylor series. Better still, our work provides an opportunity for you to challenge your students to think about how their calculators and computers "know" constants such as e. By assigning homework problems based on this paper, you can lead your students to discover for themselves new approximations to e such as ACM, MIM, those in Appendix B. . . or perhaps some we haven't discovered yet!

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APPENDIX A

Numerical Derivation of ACM

ACM was discovered by attempting to find a correlation between a new second-order approximation to e, (B1) in Appendix B, and the Classical approximation (1). Letting $f(x) \equiv (B1)$ and $g(x) \equiv$ Classical, the first step is to examine their ratio:

x	$\frac{f(x)}{g(x)}$
1	1.50000 00000
10	1.04845 38021
100	1.00498 34578
1000	1.00049 98335

[Note: since $f(x)$ at $x = 1$ is of the indeterminate form $\frac{0}{0}$, the evaluation of the ratio $\frac{f(x)}{g(x)}$ at $x = 1$ is an excellent opportunity to explore the utility of L'Hopital's Rule by comparing the analytic and numerical results.]

After subtracting 1 from this result for each generated value of x it becomes clear that the remainder of this operation times x approaches a value of 0.5 for large x:

x	$x\left[\dfrac{f(x)}{g(x)} - 1\right]$
1	0.50000 00000
10	0.48453 80210
100	0.49834 57838
1000	0.49983 34583

This suggests that

$$\lim_{x\to\infty} x\left[\frac{f(x)}{g(x)} - 1\right] = \frac{1}{2}. \tag{A1}$$

[A formal proof of (A1) is another good homework exercise!]

A new approximation with the same order of accuracy as (B1) can be obtained by substituting $h(x)$ for $f(x)$ in this expression and solving

$$x\left[\frac{h(x)}{g(x)} - 1\right] = \frac{1}{2} \tag{A2}$$

for $h(x)$. Doing so yields the closed-form expression for ACM.

APPENDIX B

Other Series-Based Approximations to *e*

The approximations below can be derived using the same bootstrapping approach outlined in the text. Their derivations are left as exercises for the reader.

$$(x+1)\left(1+\frac{1}{x}\right)^{x} - (x-1)\left(1-\frac{1}{x}\right)^{-x} \quad \text{(second order)} \tag{B1}$$

$$\frac{2x^x}{(2x-1)(x-1)^{x-1}} \quad \text{(second order)} \tag{B2}$$

$$\frac{(x+1)^{x+1}}{2x^x} + \frac{x^x}{(2x-1)(x-1)^{x-1}} - \frac{x^x}{2(x-1)^{x-1}} \quad \text{(third order)} \tag{B3}$$

These approximations can be combined with others found in the text to create the following approximations to e:

$$\frac{1}{6}ACM + \frac{5}{6}(B2) \quad \text{(third order)} \tag{B4}$$

$$\frac{8}{7}(B1) - \frac{1}{7}MIM \quad \text{(fourth order)} \tag{B5}$$

$$\frac{10}{7}(B3) - \frac{3}{7}(B4) \quad \text{(fourth order)} \tag{B6}$$

$$\frac{656}{75}(B6) - \frac{581}{75}(B5) \quad \text{(fifth order)} \tag{B7}$$